Phase Transitions in Stochastic Diffusion on a General Network

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INTRODUCTION

Can we determine the complex topology based on limited resolution measurements?

- We address this problem in the context of **Magnetic Resonance (MR)** measurements of porous structure such as **brain tissue**
- Dynamical model describing **magnetization transfer in a network of pores**
- The model is analyzed on a **general network** in the **spectral domain**
- The model can also describe population dynamics, agents problems in economy or directed polymers in random media
- It can also be interpreted as a **Kardar–Parisi–Zhang (KPZ) equation on a graph**[1]

THE MODEL

Magnetization transfer in a pore network is modelled as a Stratonovich Stochastic Differential Equations (SDEs) living on a graph under a dynamical random environment. The equations are as follows:

$$\frac{dm_i(t)}{dt} = J\sum_{j\in G} L_{ij} m_j(t) + g_i(t) m_i(t),$$

$m_i(0) = m_0$, for all i,

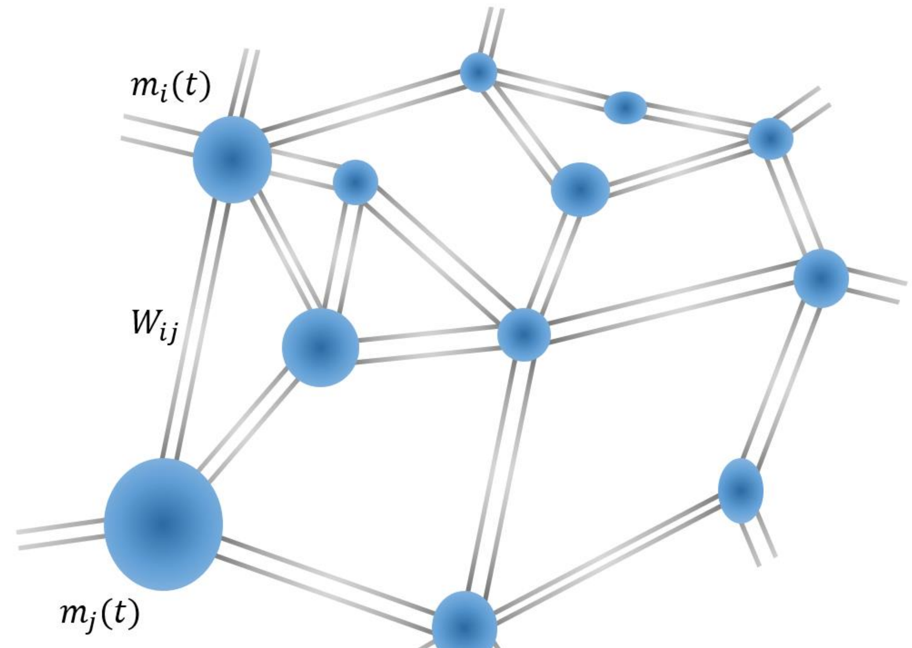
L- the graph Laplacian matrix

J - the interaction strength between pores due to diffusion of molecules between pores.

$g_i(t)$ - a multiplicative white noise,

$$\langle g_i(t)\rangle = 0, \quad \text{and} \quad \langle g_i(t) g_j(t')\rangle = \sigma^2 \delta_{ij}\delta(t-t').$$

σ^2 - the change in magnetization due to the diffusion of molecules within the pore, affected by temperature, pore geometry, and the magnetic field applied on the pore.



- Analysis of the Mean field topology shows a Pareto power-law steady state distribution and the existence of a condensation/localization phase transition[2]
- Lattice topology is known as the time dependent Parabolic Anderson Model (PAM)[3]
- Tree topology was also studied[4]

SUMMARY OF RESULTS

- Spectral generalization of the Janssen-De Dominicis technique[5]
- Perturbative calculation in σ^2 to all orders in the **spectral domain of the graph** reveals the existence of a **localization phase transition** using renormalization group arguments
- The existence of the phase transition is determined by what we define as the **collision matrix**,

$$I_{ij}(t) = P_{ij}^2(t),$$

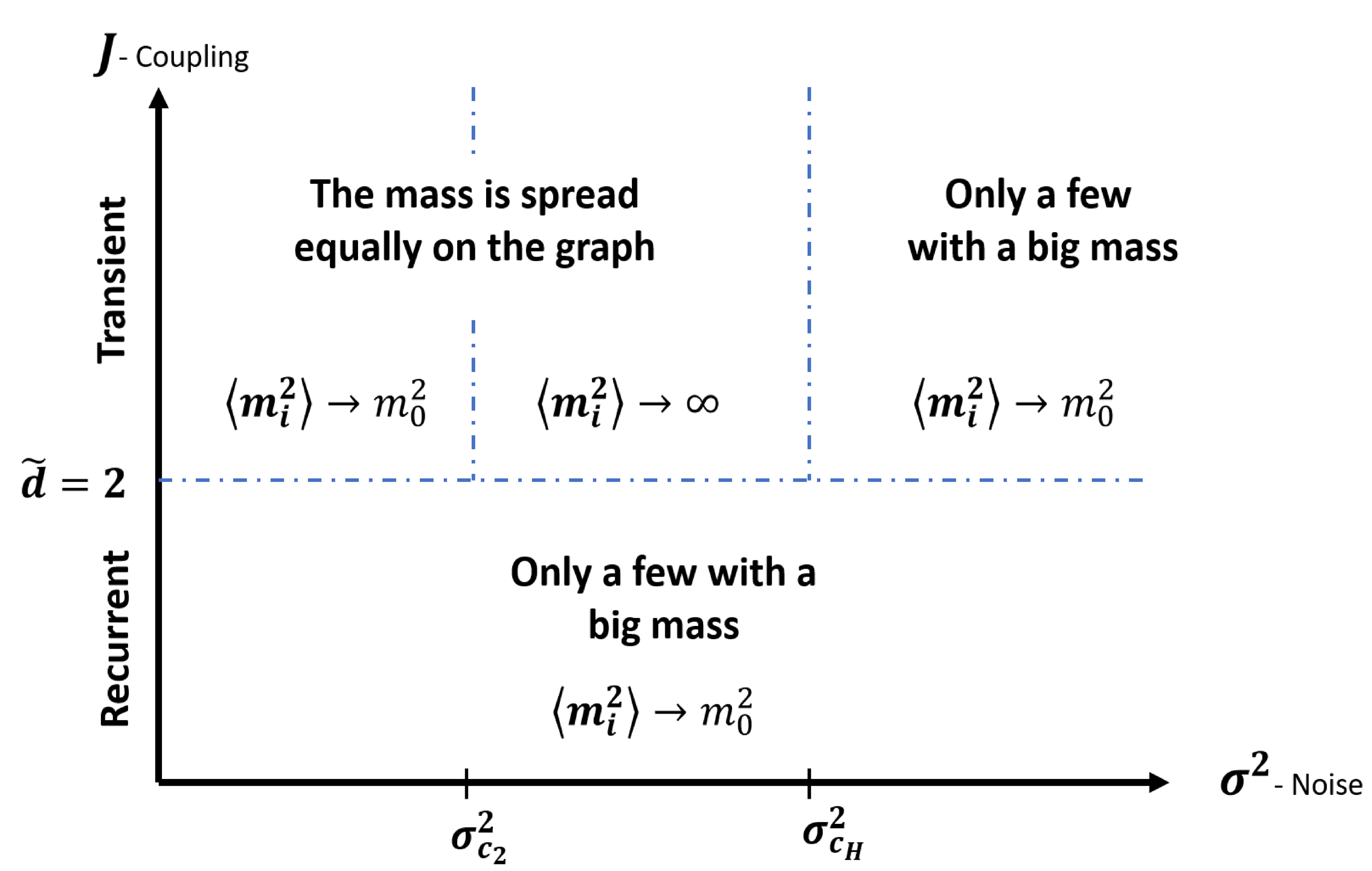
$P_{ij}(t)$ - the transition probability from site i to j at time t on the graph,

- The long time limit of this matrix is directly connected to **the probability to return to the origin** and to the **spectral dimension** (in the symmetric case):

$$\lim_{t\to\infty} \frac{\ln(\sum_k I_{ik}(t))}{\ln(t)} = \lim_{t\to\infty} \frac{\ln(P_{ii}(t))}{\ln(t)} = \frac{\tilde{d}}{2},$$

  - Measures of the complexity of the network
  - Indicates to what extent the network is **recurrent vs. transient**
  - A **measurable parameter**

- In the transient case, $\tilde{d} > 2$, there exists a localization phase transition



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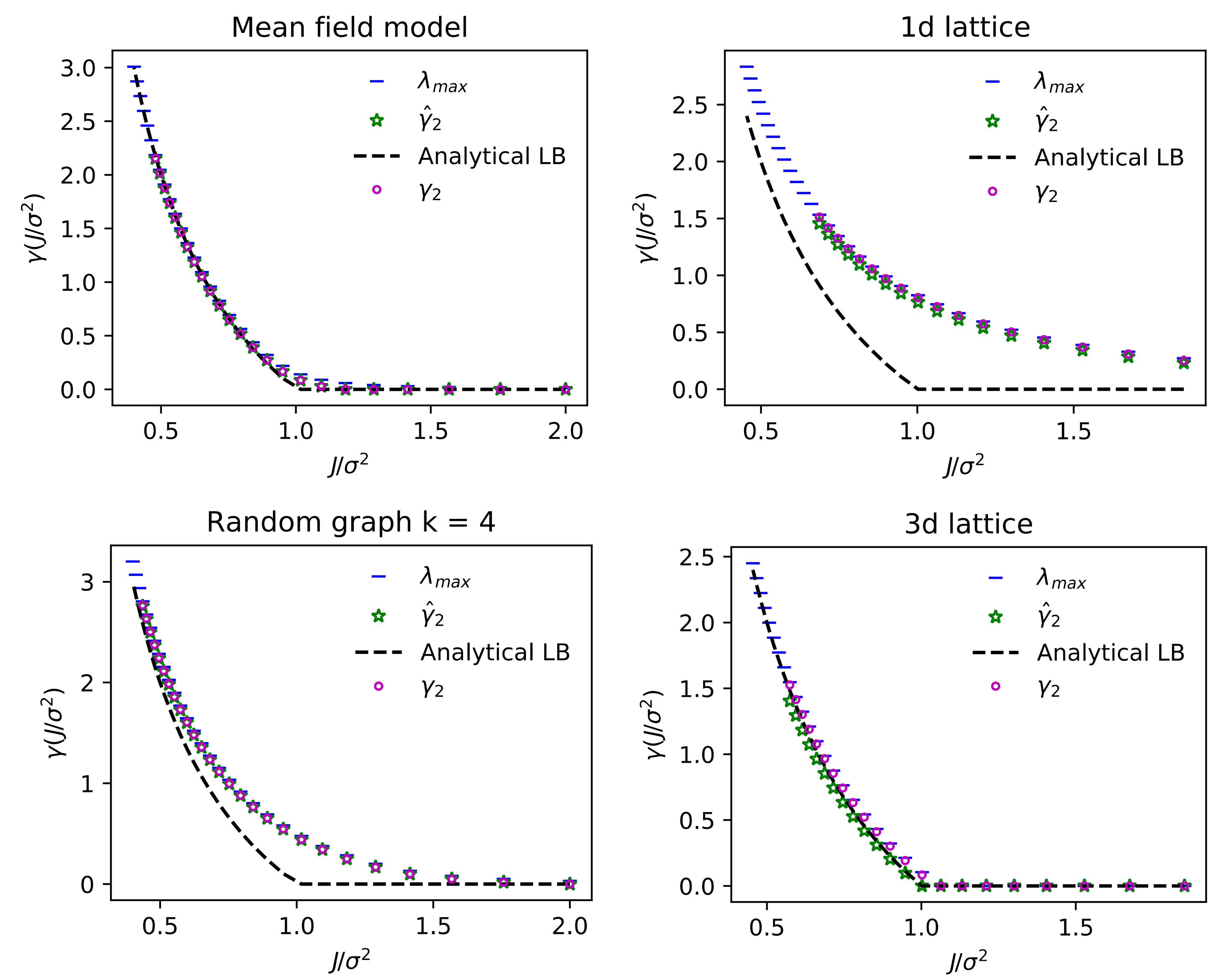
GRAPH LYAPUNOV EXPONENTS



$\hat{\gamma}_2$ - sample Lyapunov exponents for a general graph

γ_2 - moment Lyapunov exponents for a general graph

$\lambda_{\max}$ - the largest eigenvalue of the second moment equations

Analytical LB -

$$\hat{\gamma}_2 \geq 2\sigma^2 - 2J\langle k\rangle,$$

where $\langle k\rangle$ is the average degree of the graph

CONCLUSIONS

- **The complexity of the network can be extracted** by measuring the spectral dimension and the graph Lyapunov exponents
- We show **the existence of a localization phase transition**
- Using our **novel spectral technique** one can analyze other complex structured problems
- We propose to **classify tissues and brain areas**, represented as a pore network, based on their spectral dimension
- **This may lead to new experiments and observable parameters** revealing exciting structural properties of our brain as well as improvements in diagnostic of disease
- Interesting results for **time dependent random coupling and spatially varying noise**[6]