# Ergodic Effects in Token Circulation

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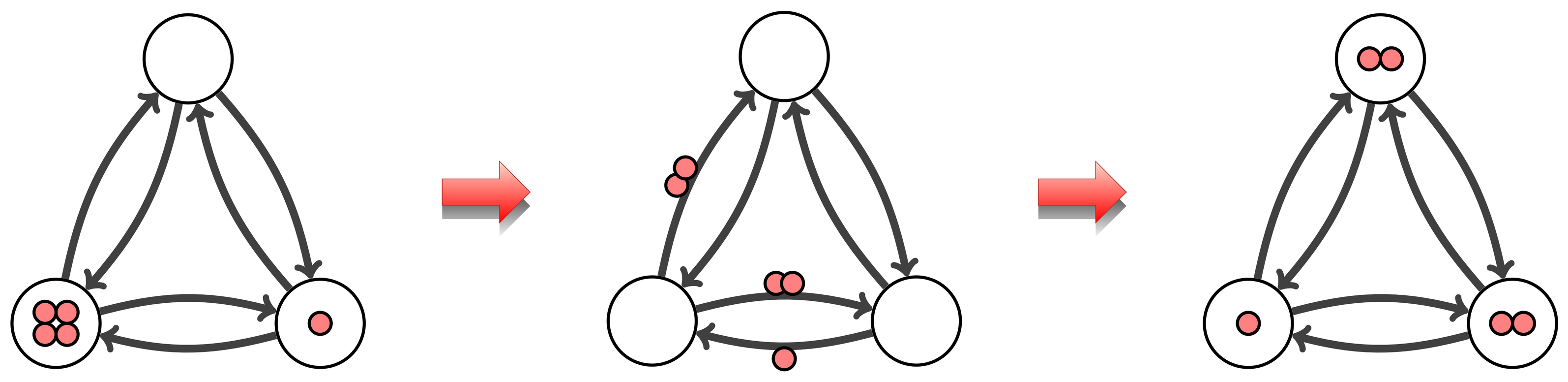
**Abstract**

We consider an *extremely simple* local rule of deterministic propagation of $k$ tokens in a graph with $m$ edges, called RR dynamics. We show that it traverses edges periodically in a highly regular manner, with $\widetilde{\Theta}(m/k)$ idle time, for a wide range of parameters, after an initial grace period. Presented at SODA 2018.

## Distributed Token Propagation



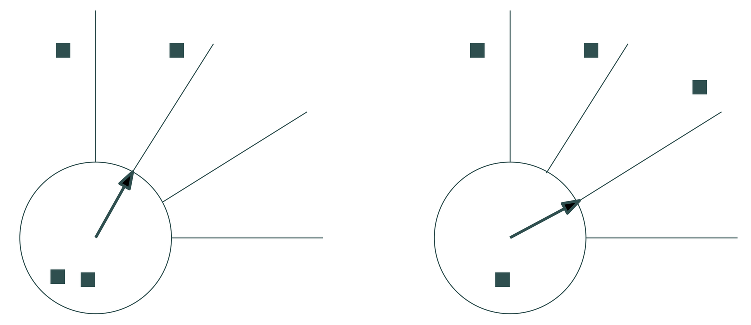
## Notation

| | | | |
|---|---|---|---|
| $n$ | total number of **vertices** | $k$ | total number of **tokens** |
| $m$ | total number of **edges** | $L_t(v)$ | **# tokens at vertex** $v$ in step $t$ |
| $2m$ | total number of **arcs** | $L_t(e)$ | # tokens **traversing arc** $e$ after step $t$ |

## RR Dynamics for Token Propagation

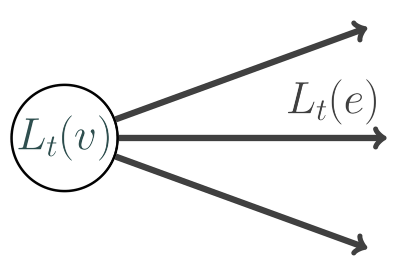
Introduced in [PDDK96]. Studied under a variety of names: *Round-robin, Rotor-router, Eulerian walker, Ant walk, Propp machine*. Each node $v$ maintains $pointer_v$ to a neighbor, following a cyclic permutation along the outgoing arcs of $v$.

**RR Dynamics:**
**While** there is at least one token at node $v$, **do**:
1. Send one token to $pointer_v$.
2. Set $pointer_v = next(pointer_v)$.



## Comparison to other Schemes

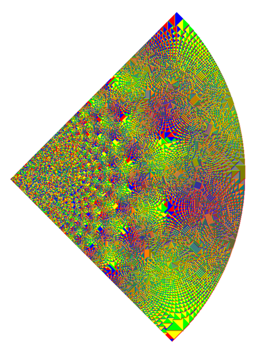
In the limit of a large number of tokens, RR dynamics follows the discrete heat diffusion equation, just like random walk diffusion.




Heat diffusion limit ($k \to \infty$): $\mathbb{E}[L_t(e)] = \frac{L_t(v)}{\deg(v)}$

RR dynamics can **compete with randomized approaches** when exploring graphs (*cover time* measure [DKPU14, KP14]) and when *load-balancing* [RSW98, DF09, BKK+15, SYKY16].

RR dynamics is a close cousin of so-called *sandpile dynamics*. In highly regular settings, such as 2-dimensional grids, it plausibly demonstrates effects of **self-organized criticality**.



## Fairness & Balancing Properties

**Local fairness bound:**

$$\left|L_t(e) - \frac{L_t(v)}{\deg(v)}\right| \le 1$$

Shared by many processes [RSW98, FS09, SS12].

**Strong fairness bound (RR only):**

$$\left|\sum_{t \le T} L_t(e) - \frac{\sum_{t\le T} L_t(v)}{\deg(v)}\right| \le 1$$

## Idle time

Deterministic token propagation processes are useful beyond load balancing. We look at long-term averaging properties. Specifically, we are interested in bounding the idle time after some *initialization time* $T_{\text{init}}$.

**Definition.** Idle time *of a token propagation scheme is the smallest value $T$ such that in any consecutive $T$ steps every edge/arc is traversed at least once.*

## References

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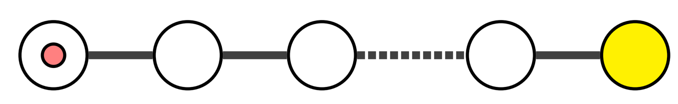
## Issue: Enforcing Regular & Periodic Behavior

Random-walk type processes tend to behave poorly w.r.t. idle time.



For example, **idle time** of a path (with a single token) is $\Omega(n^2)$ for the random walk, but $\Theta(n)$ for RR dynamics.

## Main result

**Theorem.** *For any time $t \ge T_{init} = poly(n, \log k)$, the idle time of RR satisfies the following bounds:*
- $\widetilde{\mathcal{O}}(\gcd(k, 2m)\frac{m}{k})$
- $\widetilde{\mathcal{O}}(\sqrt{n} \cdot \frac{m}{k})$
- $\mathcal{O}(\frac{m}{k}) = \mathcal{O}(1)$ *for* $k \ge (\frac{1}{2} + \varepsilon)m$
- $\widetilde{\mathcal{O}}(\sqrt{k} \cdot \frac{m}{k})$
- $\mathcal{O}(Diam \cdot \frac{m}{k})$
- $\mathcal{O}(\frac{m}{k})$ *for trees*

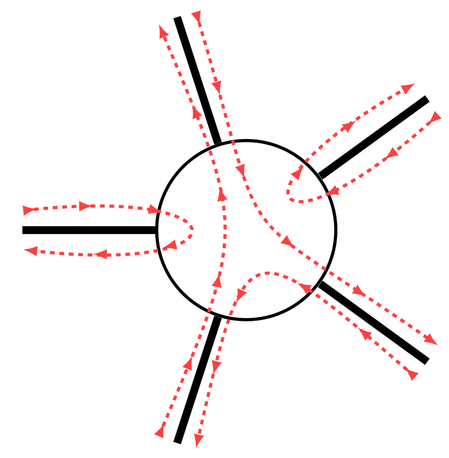
The RR dynamics has almost-optimal idle time $\widetilde{\mathcal{O}}(\frac{m}{k})$ when $m$ and $k$ are co-prime.

## Techniques

### Eulerian circulation

**Theorem** ([CDG+15]). *Recurrent state of RR $\Leftrightarrow$ there is a bijection $\varphi : \vec{E} \to \vec{E}$ such that*
- *$\varphi(e)$ always starts where $e$ ends*
- *$L_{t+1}(\varphi(e)) = L_t(e)$*

*Recurrent state is reached in time $poly(n, \log k)$.*



### The gcd connection

$$\lim_{T\to\infty} \frac{\sum_{t\le T} L_t(e)}{T} = \frac{k}{2m}$$

therefore

$$\frac{\text{\# tokens on a cycle}}{\text{\# arcs on a cycle}} = \frac{k}{2m}$$

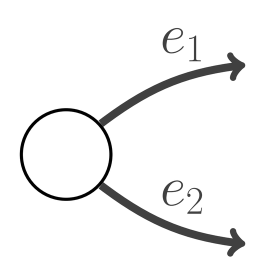
#cycles | $\gcd(k, 2m)$

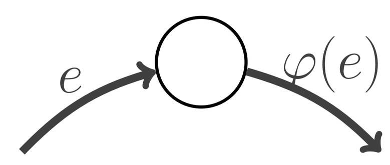
$\gcd(k, 2m) = 1 \implies$ single cycle

### Skewed distances

$\delta_0(e_1, e_2) = 1$

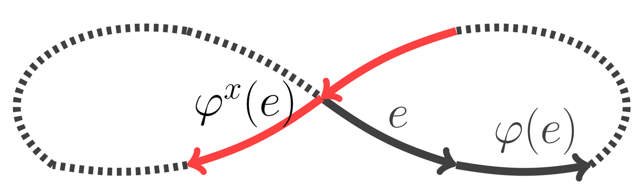
$\delta_1(e, \varphi(e)) = 0$




$$\delta_{\Delta t}(e, e') \stackrel{\text{def}}{=} \max_{t_1,t_2} \left|\sum_{t_1 \le t \le t_2} \left(L_{t+\Delta t}(e_1) - L_t(e')\right)\right|$$

### Self-intersections




**Definition.** *$x$ is a **self-intersection** of the cycle: for some $e$, $e$ and $\varphi^x(e)$ share starting vertex.*

**Discrepancy parameter** $\delta$:

$$\delta_x \stackrel{\text{def}}{=} \delta_x(e, e) = \delta_0(e, \varphi^x(e))$$

- $\delta_0 = 0$
- $\delta_x \le 1$ for $x$ self-intersections
- $\delta_{x+y} \le \delta_x + \delta_y$

$$\left|\sum_{t\le T} L_t(e) - \frac{k}{2m}T\right| \ge \max_{0\le x<2m} \delta_x$$

**Idle time**$(e) = \mathcal{O}\left(\frac{m}{k}\right) \cdot \max_{0\le x<2m} \delta_x$

### Additive combinatorics

**Sumset**: For $A, B \subseteq \mathbb{Z}_{2m}$, $A + B \stackrel{\text{def}}{=} \{a + b : a \in A, b \in B\}$.

**Multiplication**: $\kappa \cdot A \stackrel{\text{def}}{=} \underbrace{A + A + \ldots + A}_{\kappa \text{ times}}$.

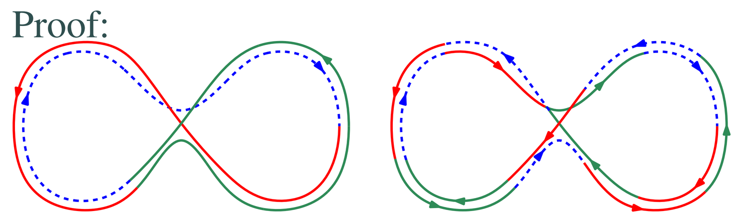
$\mathcal{X}$ is the set of all self-intersections. What is the smallest $\kappa$, such that $\kappa \cdot \mathcal{X} = \mathbb{Z}_{2m}$?

### Spectral property

**Lemma.**

$$\forall_{f\ge 1} \quad \exists_{x\in\mathcal{X}} \quad (f \cdot x) \in \left[\frac{2}{3}m, \frac{4}{3}m\right]$$

Proof:



### Bohr sets

**Restating:** $\forall_{f\ge1} \quad \exists_{x\in\mathcal{X}} \quad (f \cdot x) \in \left[\frac{2}{3}m, \frac{4}{3}m\right] \quad \Leftrightarrow \quad \text{Bohr}(\mathcal{X}, 1/6) = \{0\}$

**Theorem.** *For any $A \subseteq \mathbb{Z}_{2m}$, if $Bohr(A, 1/6) = \{0\}$, then $\kappa \cdot X = \mathbb{Z}_{2m}$ for some $\kappa = \mathcal{O}(\log^2 m)$.*

**Proof**: [main technical contribution, generalizes [TV06] Proposition 4.40]

$\Downarrow$

$$\max_{0\le x<2m} \delta_x = \mathcal{O}(\log^2 m)$$