# Storage capacity in symmetric binary perceptrons

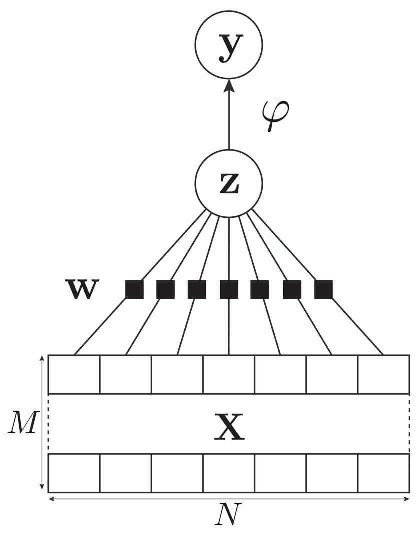
Benjamin Aubin †, Will Perkins ⋆, Lenka Zdeborová†
†*Institut de physique théorique, Université Paris Saclay, France*
⋆*University of Birmingham, England*

## Summary

We reviewed the problem of computing the storage capacity of binary perceptron [1, 2] with random patterns. The perceptron problem is motivated by studies of simple artificial neural networks and we view it here as a random constraint satisfaction problem (CSP) where the vector of binary weights $\mathbf{w} \in \{\pm 1\}^N$ must satisfy $M$ constraints. For the *step* constraint, the storage capacity was predicted in [1, 2] using replica method, but proving this result stays up to date an **open mathematical problem**. We consider instead **symmetric constraints** and **compute the capacity** $\alpha_c$ **rigorously**.

## The simplest neural network

- $K \in \mathbb{R}$ a threshold / stability
- $X_{\mu i} \sim \mathcal{N}\left(0, \frac{1}{N}\right)$ i.i.d
- $\mathbf{X} \in \mathbb{R}^{M \times N}$
- $M$ patterns $\mathbf{X}_{\mu=1\dots M}$
- Binary weights $\mathbf{w} \in \{\pm 1\}^N$
- Constraint density $\boxed{\alpha \equiv M/N}$




The perceptron *satisfies the pattern* $\mu$ if the vector $\mathbf{w}$ satisfies the constraint:
$$\mathbf{X}_\mu \mathbf{w} \geq K$$

## Storage capacity problem

- Number of satisfying vectors: $\mathcal{Z}(\mathbf{X}) = \sum_{\mathbf{w} \in \{\pm 1\}^N} \mathbb{1}(\mathbf{w} \text{ satisfies all patterns } \mathbf{X}_\mu)$
- Free entropy: $\phi(\alpha) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\mathbf{X}}[\log(\mathcal{Z}(\mathbf{X}))]$

For a general constraint function $\varphi$, the **storage capacity** $\alpha_c(K)$ is defined as the largest value of $\alpha$ such that in the limit $N \to \infty$ and $\alpha = \mathcal{O}(1)$, all the constraints are simultaneously satisfiable with high probability (satisfiability threshold):

$$\alpha_c(K) = \inf\left\{\alpha : \lim_{N\to\infty} \mathbb{P}\left[\ \exists \mathbf{w}\ /\ \forall \mu \in [1:M],\ \varphi(\mathbf{X}_\mu \mathbf{w}) = 1\ \right] = 0\right\} \overset{?}{=} \sup\left\{\alpha : \lim_{N\to\infty} \mathbb{P}\left[\ \exists \mathbf{w}\ /\ \forall \mu,\ \varphi(\mathbf{X}_\mu \mathbf{w}) = 1\ \right] = 1\right\}$$

- ▶ For the binary perceptron, the capacity is computed when the RS free entropy vanishes $\phi(\alpha_c) = 0$ [2] $\longrightarrow \alpha_c^s(K=0) \simeq 0.833$, but proving this result remains a challenging mathematical problem
- ▶ We analyze **symmetric variants** of the classical step **binary perceptron**: rectangle and symmetric step

## Definitions

Let $z, z' \sim \mathcal{N}(0,1)$ *i.i.d.* Let $\beta \in [0;1]$, $z_1 = \sqrt{\beta} z + \sqrt{1-\beta} z'$ and $z_2 = \sqrt{\beta} z - \sqrt{1-\beta} z'$

- $p_K^{\mathbf{s},\mathbf{r},\mathbf{u}} \equiv \mathbb{P}[\varphi^{\mathbf{s},\mathbf{r},\mathbf{u}}(z) = 1]$
- $q_K^{\mathbf{r},\mathbf{u}}(\beta) \equiv \mathbb{P}[\varphi^{\mathbf{r},\mathbf{u}}(z_1) = 1 \wedge \varphi^{\mathbf{r},\mathbf{u}}(z_2) = 1]$
- $H(\beta) \equiv -\beta \log(\beta) - (1-\beta)\log(1-\beta)$
- $F_{K,\alpha}^{\mathbf{r},\mathbf{u}}(\beta) \equiv H(\beta) + \alpha \log(q_K^{\mathbf{r},\mathbf{u}}(\beta))$

## Theorem

**Theorem 1.** *Under the following assumption: for all choices of $K > 0$ and $\alpha > 0$ so that $\partial_\beta^2 F_{K,\alpha}^{\mathbf{r},\mathbf{u}}(1/2) < 0$, there is exactly one $\beta \in (1/2, 1)$ so that $\partial_\beta F_{K,\alpha}^{\mathbf{r},\mathbf{u}}(\beta) = 0$:*

1. *For all $K > 0$, we have $\alpha_c^{\mathbf{r}}(K) = -\log(2)/\log(p_K^{\mathbf{r}})$.*
2. *For all $K \in (0, K^* \simeq 0.817)$, we have $\alpha_c^{\mathbf{u}}(K) = -\log(2)/\log(p_K^{\mathbf{u}})$.*

## $1^{st}$ moment

**Proposition 2.** *If $\alpha > \alpha_a^{\mathbf{s},\mathbf{r},\mathbf{u}}(K) = \frac{-\log(2)}{\log(p_K^{\mathbf{s},\mathbf{r},\mathbf{u}})}$, with high probability (whp) there is no satisfying assignment to the binary perceptron with the $\{\mathbf{s},\mathbf{r},\mathbf{u}\}$ activation functions.*

*Proof.* $\mathbb{P}[\mathcal{Z}_{\mathbf{s},\mathbf{r},\mathbf{u}}(\mathbf{X}) > 0] \leq \mathbb{E}[\mathcal{Z}_{\mathbf{s},\mathbf{r},\mathbf{u}}(\mathbf{X})] = 2^N \mathbb{E}\left[\prod_{\mu=1}^M \varphi_{\mathbf{s},\mathbf{r},\mathbf{u}}(z_\mu(\mathbf{1}))\right] = 2^N p_K^{\mathbf{s},\mathbf{r},\mathbf{u}} = \exp(N(\log(2) + \alpha \log(p_K^{\mathbf{s},\mathbf{r},\mathbf{u}}))) \to 0$ as $N \to \infty$ □

## $2^{nd}$ moment

**Proposition 3.** 1. $\forall K > 0$, $\forall \alpha < \alpha_a^{\mathbf{r}}(K)$, $\exists c_2 > 0$ *s.t* $\mathbb{P}[\mathcal{Z}_{\mathbf{r}}(\mathbf{X}) > 0] \geq \frac{1}{c_2} > 0$

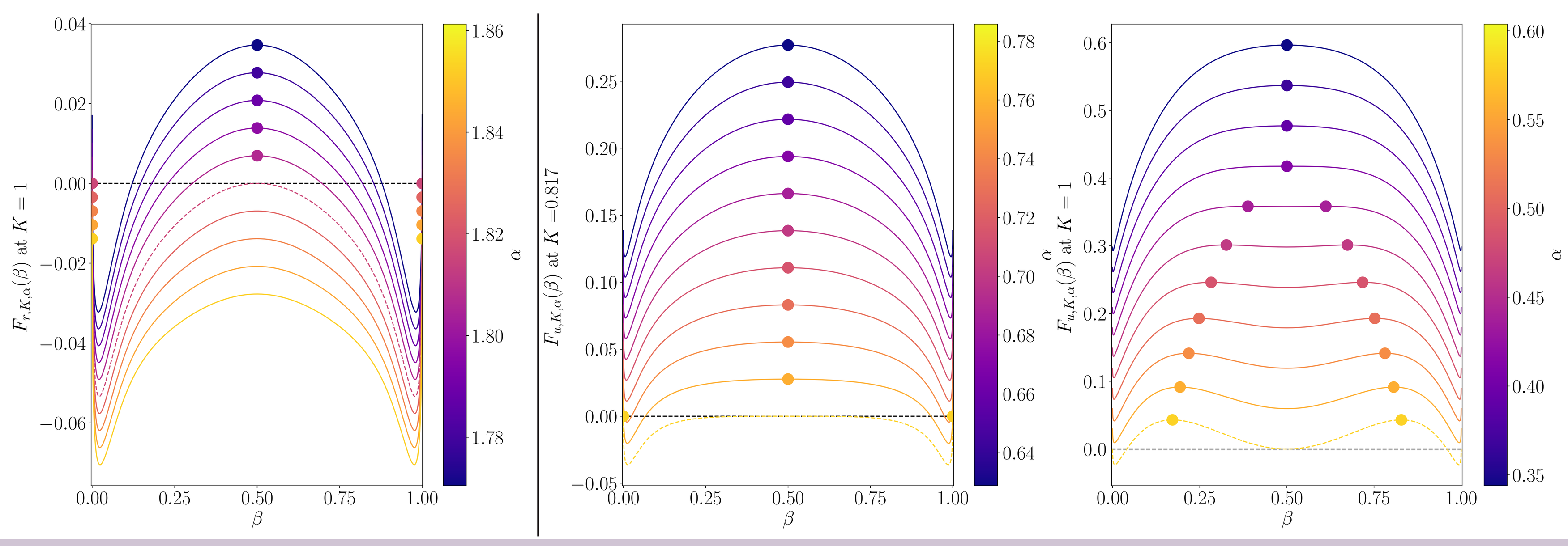
2. $\forall K \in (0, K^*)$, $\forall \alpha < \alpha_a^{\mathbf{u}}(K)$, $\exists c_2 > 0$ *s.t* $\mathbb{P}[\mathcal{Z}_{\mathbf{u}}(\mathbf{X}) > 0] \geq \frac{1}{c_2} > 0$

*Proof.* As $\mathcal{Z}_{\mathbf{r},\mathbf{u}}(\mathbf{X}) \in \mathbb{N}$, $\mathbb{P}[\mathcal{Z}_{\mathbf{r},\mathbf{u}}(\mathbf{X}) > 0] \geq \frac{\mathbb{E}[\mathcal{Z}_{\mathbf{r},\mathbf{u}}(\mathbf{X})]^2}{\mathbb{E}[\mathcal{Z}_{\mathbf{r},\mathbf{u}}(\mathbf{X})^2]}$

- $\mathbb{E}[\mathcal{Z}_{\mathbf{r},\mathbf{u}}(\mathbf{X})]^2 = \left(2^N p_K^{\mathbf{r},\mathbf{u}}\right)^2$
- If the supremum is achieved for $\beta = 1/2$, using Laplace's method, $\exists c_2 > 0$ s.t:

$$\mathbb{E}[\mathcal{Z}_{\mathbf{r},\mathbf{u}}^2(\mathbf{X})] \leq 2^N e^{N \sup_{\beta \in [0;1]}\{F_{K,\alpha}^{\mathbf{r},\mathbf{u}}(\beta)\}} \leq c_2 2^N e^{N\left(\{H(1/2) + \alpha \log(q_K^{\mathbf{r},\mathbf{u}}(1/2))\}\right)} = c_2 4^N (p_K^{\mathbf{r},\mathbf{u}})^{2\alpha N}$$

- $\mathbf{u}$: $\forall K, \forall \alpha < \alpha_a^r(K)$, $\beta^* = 1/2$ || $\mathbf{r}$: $\forall K \leq K^*$, $\forall \alpha < \alpha_a^r(K)$, $\beta^* = 1/2$. Wrong for $K \geq K^*$: $2^{nd}$ method fails - onset of the RSB region.




## Conclusion and summary of the results

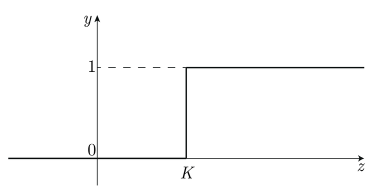
While the storage capacity for the step binary perceptron has not been proven yet, we provided a rigorous proof of the capacity for similar symmetric constraints (rectangle and symmetric step).

| | Constraint | Range of $K$ | Storage capacity |
|---|---|---|---|
| Step function | $z \geq K$ , $\varphi^s$ | $\forall K \in \mathbb{R}$ | RS |
| Rectangle | $\lvert z\rvert \leq K$ , $\varphi^r$ | $\forall K \in \mathbb{R}^+$ | Annealed |
| Symmetric step | $\lvert z\rvert \geq K$, $\varphi^u$ | $0 < K < K^* = 0.817$ | Annealed |
| Symmetric step | $\lvert z\rvert \geq K$, $\varphi^u$ | $\forall K > K^* = 0.817$ | FRSB(?) |

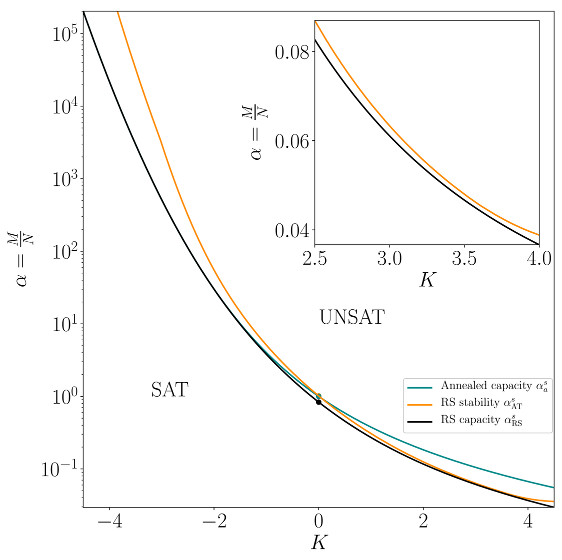
## Main results and plots

◇ **Step - s**

$\boxed{\varphi^s(z) \equiv \mathbb{1}(z \geq K)}$

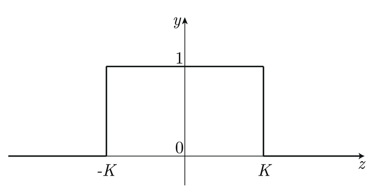


- ⋆ RS solution stable $\forall K \in \mathbb{R}$
- ⋆ $\alpha_c^{\mathbf{s}} = \alpha_{RS}^{\mathbf{s}} < \alpha_a^{\mathbf{s}}$ for $K \in \mathbb{R}$ but **not proven rigorously**, second moment method fails...!
- ⋆ $\alpha_c^{\mathbf{s}}(0) \simeq 0.833$, $\alpha_{AT}^{\mathbf{s}}(0) = 1.015$
- ⋆ Configuration space: *frozen* 1RSB

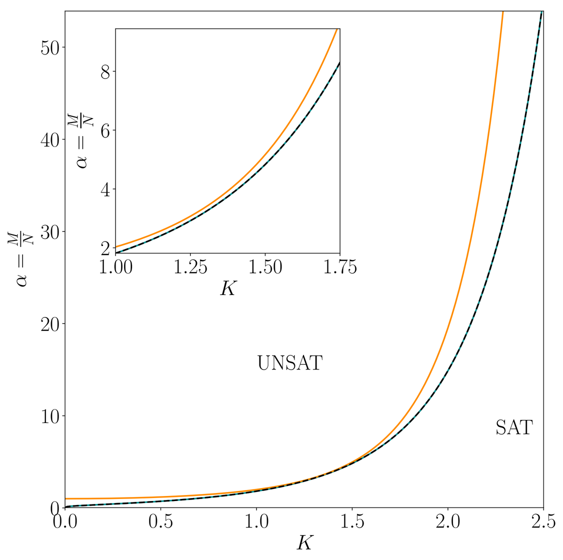



◇ **Rectangle - r**

$\boxed{\varphi^r(z) \equiv \mathbb{1}(|z| \leq K)}$

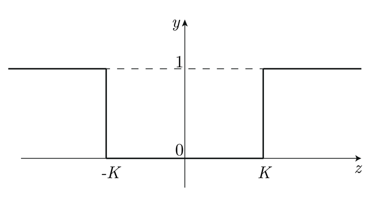


- ⋆ RS solution stable $\forall K \in \mathbb{R}^+$
- ⋆ Annealed entropy matches the RS entropy: $\phi_a = \phi_{RS}$
- ⋆ $\forall K > 0$, $\alpha_c^{\mathbf{r}}(K) = \alpha_a^{\mathbf{r}}(K) = \alpha_{RS}^{\mathbf{r}}(K)$: by $2^{nd}$ moment method
- ⋆ Configuration space: *frozen* 1RSB

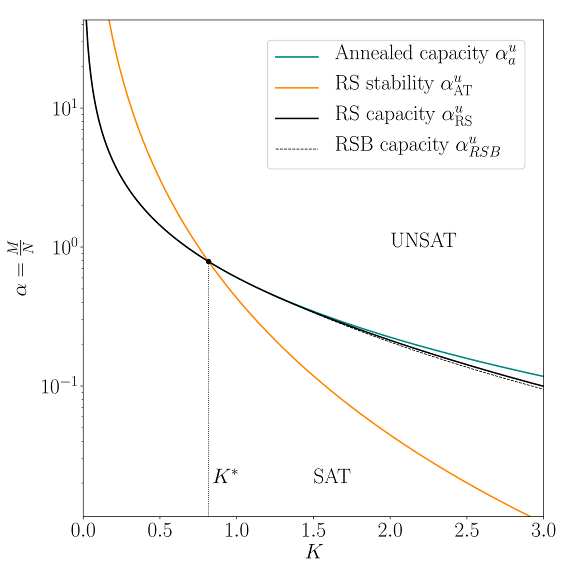



◇ **Symmetric step - u**

$\boxed{\varphi^u(z) \equiv \mathbb{1}(|z| \geq K)}$

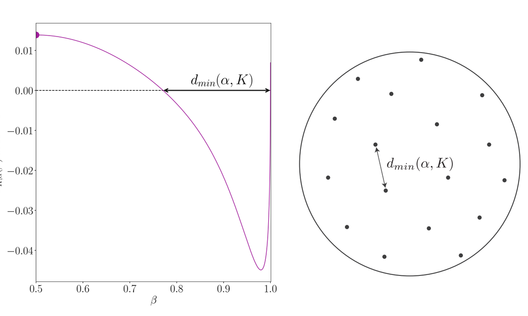


- For $K < K^* \simeq 0.817$
  - ⋆ RS solution stable
  - ⋆ Annealed entropy matches the RS entropy: $\phi_a = \phi_{RS}$
  - ⋆ $\alpha_c^{\mathbf{u}}(K) = \alpha_a^{\mathbf{u}}(K) = \alpha_{RS}^{\mathbf{u}}(K)$: by $2^{nd}$ moment method
  - ⋆ Configuration space: *frozen* 1RSB
- For $K \geq K^*$
  - ⋆ RS solution becomes unstable at $K = K^*$
  - ⋆ $2^{nd}$ moment method fails exactly at the same value $K^*$ (AT line / RS capacity)
  - ⋆ 1RSB solution unstable ⟶ Full RSB ?




## Frozen 1RSB

- $\partial_\beta F_{K,\alpha}^{\mathbf{r},\mathbf{u}}|_{\beta=1} = +\infty$
- $\forall K, \alpha, \exists d_{min}(\alpha, K)$ : $\forall d \in ]0; d_{min}[$, $\forall \mathbf{w}_1, \mathbf{w}_2 \in \{\pm 1\}^N$ SAT, $\mathbb{P}[\frac{1}{N}\mathbf{w}_1^T \mathbf{w}_2 = 1 - d] = 0$

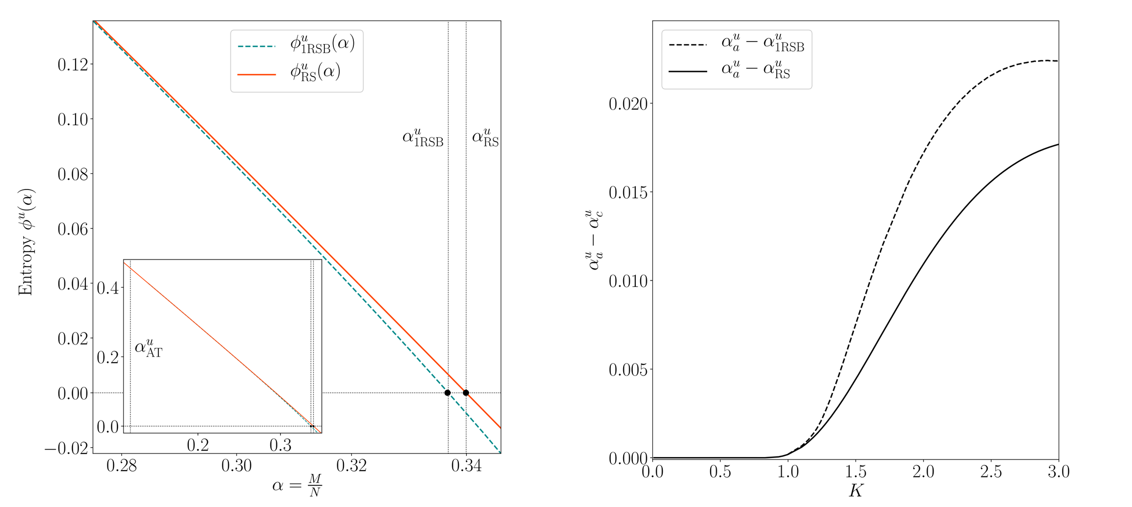


- The configuration space is *frozen*: clusters of single solutions at a distance $d \geq d_{min}$
- True for $\mathbf{s}$, $\mathbf{r}$ and $\mathbf{u}$

## Replicas

$$\phi(\alpha) = \lim_{N\to+\infty} \lim_{n\to 0} \frac{1}{Nn} \frac{\partial \log\left(\mathbb{E}_{\mathbf{X}}[\mathcal{Z}(\mathbf{X})^n]\right)}{\partial n}$$

- RS solution stable for $K < K^*$, where $K^*$ is the intersection between AT-line and annealed capacity
- 1RSB capacity provides a small correction
- 1RSB solution unstable ($q_0 \neq 0$ while symmetry should impose $q_0 = 0$)
- Probably Full RSB (analysis to be done)




## References

[1] E. Gardner & B. Derrida. Optimal storage properties of neural network models. *J. Phys. A: Math. and Gen*, 1988.
[2] W. Krauth & M. Mézard. Storage capacity of memory networks with binary couplings. *J. Phys. France*, 1989.