The Random Fractional Matching Problem

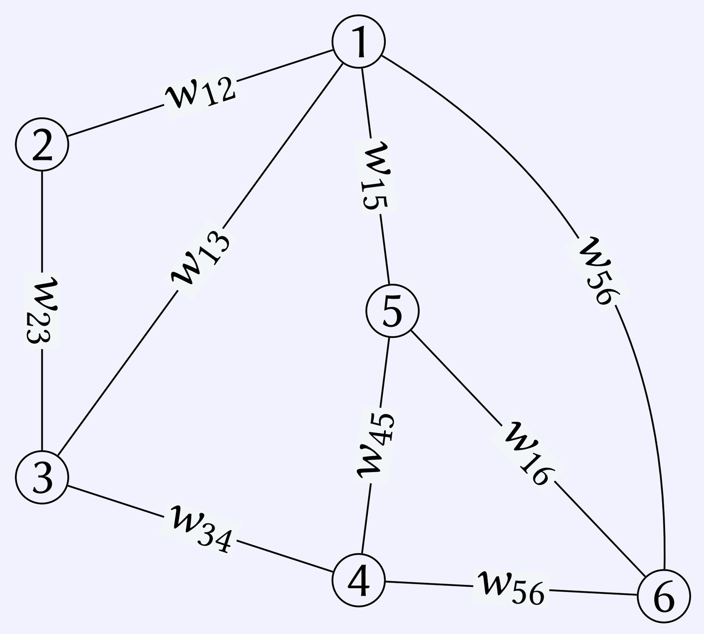
Allowing loops and cycles in random matching problems

J. Stat. Mech. (2018) 053301

Carlo Lucibello[1] Enrico Malatesta[2] Giorgio Parisi[3] **Gabriele Sicuro**[3]

[1]Politecnico di Torino and Human Genetics Foundation, Turin, Italy
[2]University of Milan, Milan, Italy [3]Sapienza University of Rome, Rome, Italy

Simons Collaboration on Cracking the Glass Problem



Input We consider a graph $\mathcal{G}$ and we associate to each edge e of a graph $\mathcal{G}$ a **random weight** $w_e \geq 0$ extracted from a certain probability density $\rho(w)$.

Variables We associate to each edge of a graph $\mathcal{G}$ a variable $m_e \geq 0$ such that for each vertex i of the graph

$$\sum_{e\to i} m_e = 1 \quad \text{with} \quad \begin{cases} m_e \in \{0,1\} & \textbf{integer problem}\text{: no cycles in the solution} \\ 0 \leq m_e \leq 1 & \textbf{fractional problem}\text{: solutions with cycles allowed} \end{cases}$$

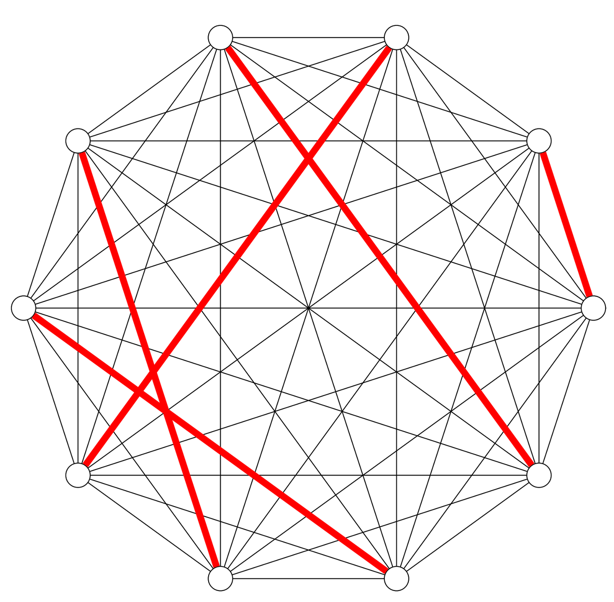
Target We search for the set $m^* = \{m_e^*\}_e$ that satisfies the above constraints and that **minimizes**

$$E[m^*] = \sum_e m_e^* w_e := \min_m \sum_e m_e w_e.$$

The question

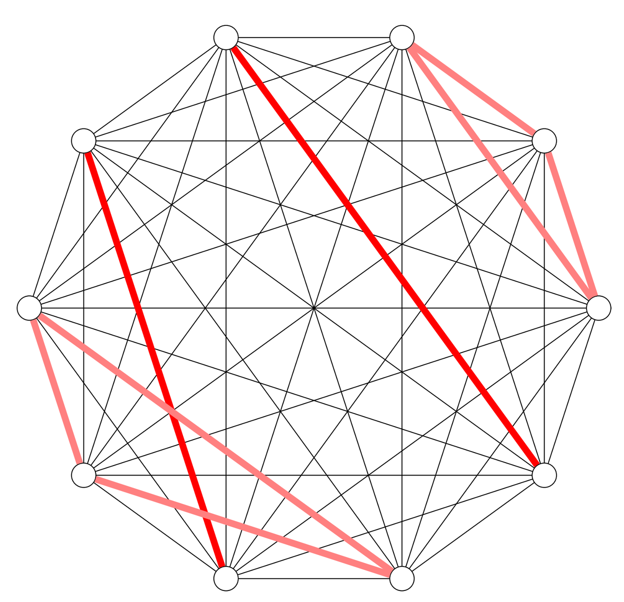
Given a class of graphs and a distribution ρ, what is the **average** over all instances of the optimal cost $\overline{E[m^*]}$?
We try to answer working on **complete graphs with** $2N$ **vertices** for $N \gg 1$ and $\rho(w) = e^{-w} = 1 - w + o(w)$.

Matching



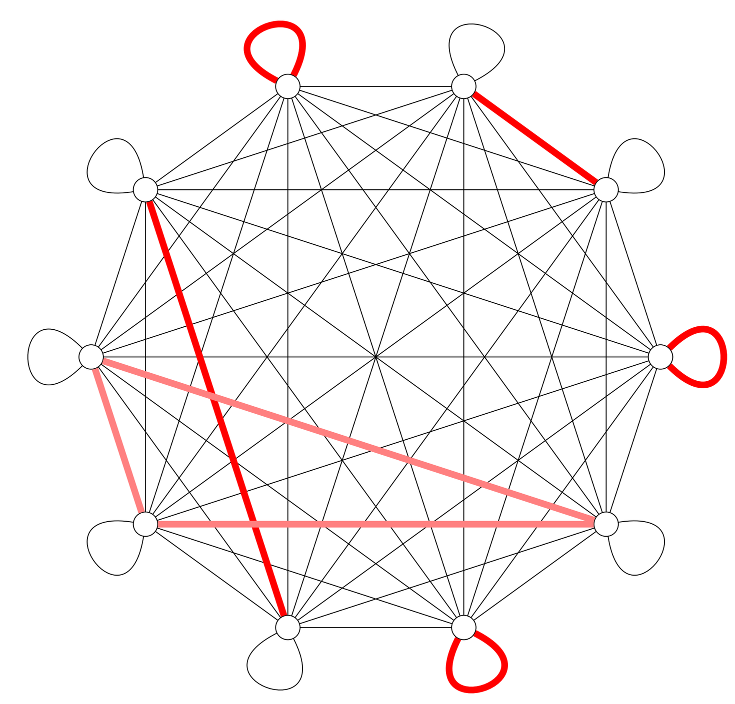
By hypothesis, $m_e \in \{0,1\}$. Cycles are not allowed in the solution.

Fractional Matching



By hypothesis, $0 \leq m_e \leq 1$, but it can be proved that in the optimal solution $m_e \in \{0, 1/2, 1\}$. Cycles may appear of length $\ell = 2k+1$ with $k \in \mathbb{N}$.

"Loopy" Fractional Matching



By hypothesis, $0 \leq m_e \leq 1$, but it can be proved that in the optimal solution $m_e \in \{0, 1/2, 1\}$. Cycles may appear of length $\ell = 2k+1$ with $k \in \mathbb{N}_0$.

Asymptotic properties

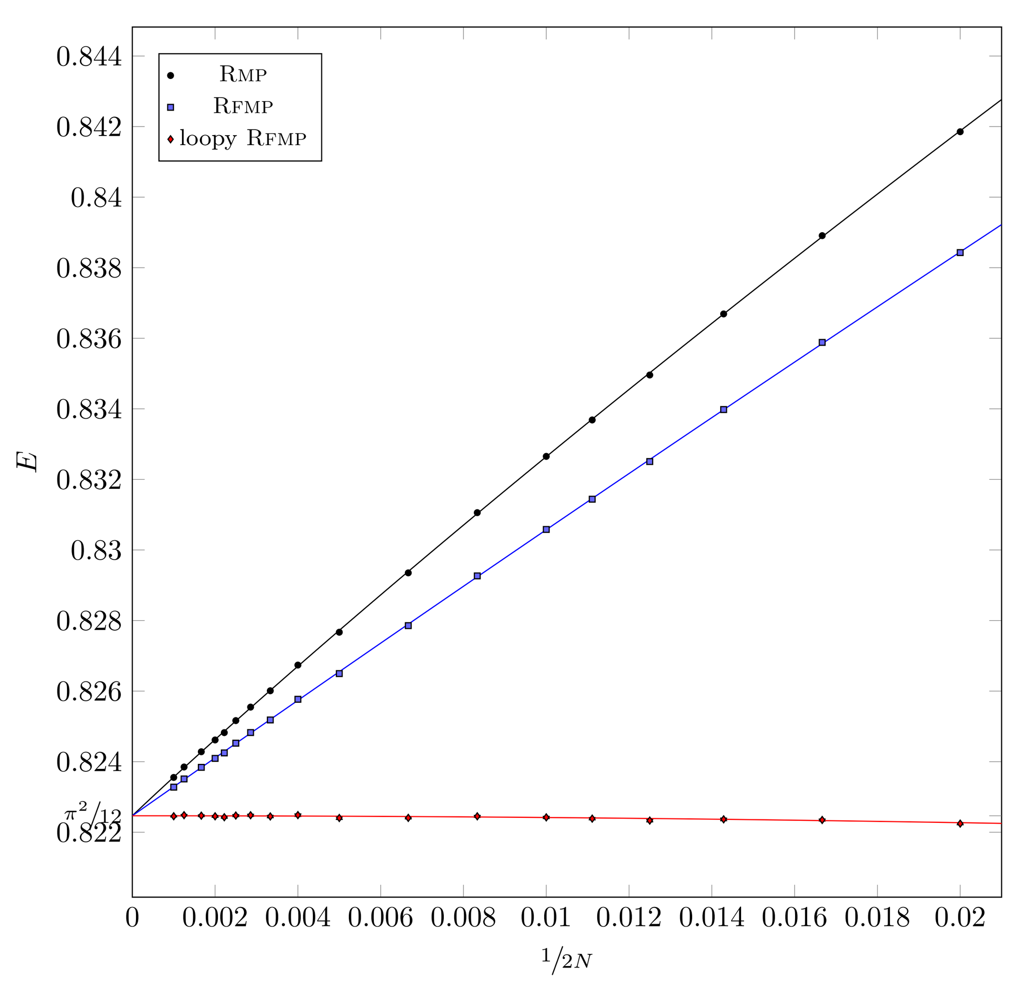
The three models have the **same asymptotic properties** that can be calculated using the **replica trick**. In particular, in all the three cases

$$Z(\beta) = \sum_{m \text{ suitable}} e^{-2\beta N E[m]} \Rightarrow \boxed{\overline{E[m^*]} = \lim_{\beta\to+\infty} \lim_{\substack{n\to 0\\ N\to+\infty}} \frac{1-\overline{Z^n(\beta)}}{n\beta N} = \frac{\zeta(2)}{2}}$$

[Mézard, Parisi (1985)] and

$$\boxed{\overline{\left(E[m^*] - \overline{E[m^*]}\right)^2} = \frac{\zeta(2)-\zeta(3)}{N}}$$

[new].



The fact that cycles and loops are allowed or not **strongly affects** the structure of the finite-size corrections.
In the **matching problem** it is known that, for a certain operator T and a constant γ

$$\overline{E[m^*]} = \frac{\zeta(2)}{2} + \frac{1}{2N}\left(\frac{\zeta(2)}{4} + \sum_{k=1}^{\infty} \frac{\operatorname{Tr}\left[\mathsf{T}^k\right]}{2k+1} + \frac{\gamma}{\sqrt{N}}\right) + O\left(\frac{1}{N^2}\right).$$

We found that in the **random fractional matching problem**

$$\overline{E[m^*]} = \frac{\zeta(2)}{2} + \frac{1}{2N}\left(\frac{\zeta(2)}{4}\right) + O\left(\frac{1}{N^2}\right)$$

whereas in the **"loopy" random fractional matching problem**

$$\overline{E[m^*]} = \frac{\zeta(2)}{2} + O\left(\frac{1}{N^2}\right)$$

(in this case exact results are available due to Wästlund).

The suppression of cycles and loops in the optimal solution **strongly** affects the finite-size corrections in random matching problems. In the standard random matching problem, in particular, a correction appears that *looks like* a cycles contribution, alongside with a finite-size correction to it, i.e., the $\gamma/\sqrt{N}$ term. As soon as cycles are allowed, the correction disappears. The remaining term can be removed simply allowing loops, i.e., "loops of length 1". As it happens in the study of disordered systems, the finite-size correction depends on topological structures of the underlying graph.

Seminal paper M. Mézard, G. Parisi, J. Phys. Lett. 46, 771 (1985).
Finite size corrections in the matching problem M. Mézard, G. Parisi, J. Phys. 48, 1451 (1987); G. Parisi, M. Ratiéville, Eur. Phys. J. B 29, 457 (2002); S. Caracciolo, M.P. D'Achille, E. Malatesta, G. Sicuro, arXiv:1702.05991 (2017).
Rigorous results on the "loopy" fractional matching J. Wästlund, Acta Mathematica 204, 91–150 (2010).

gabriele.sicuro@roma1.infn.it